On $\mathfrak{sl}(2)$-equivariant quantizations

S. Bouarroudj [*]   M. Iyadh Ayari [†]

Abstract

By computing certain cohomology of Vect(M) of smooth vector fields we prove that on 1-dimensional manifolds M there is no quantization map intertwining the action of non-projective embeddings of the Lie algebra $\mathfrak{sl}(2)$ into the Lie algebra Vect(M). Contrariwise, for projective embeddings $\mathfrak{sl}(2)$-equivariant quantization exists.

1   Introduction

Quantization, according to one of many definitions, is a procedure which to each polynomial on a symplectic manifold (in our case it is T^*M) assigns a differential operator on a Hilbert space (see, e.g., [25]). An *equivariant* (with respect to a finite-dimensional Lie subalgebra $\mathfrak{g} \subset$ Vect(M) of the Lie algebra of all smooth vector fields on M) quantization intertwines a local action of $\mathfrak{g} \subset$ Vect(M), see [7, 8, 19]. For $M = \mathbb{R}$ or S^1, this point of view was initiated in [5] in the context of a Lie subalgebra $\mathfrak{sl}(2) \subset$ Vect(M). In fact, the Lie algebra $\mathfrak{sl}(2)$ has various embeddings into Vect(M) (for $M = \mathbb{R}$, only $n = 0$ is possible):

$$\mathfrak{l}_n = \text{Span}\left(x^{-n}\frac{d}{dx}, \quad x\frac{d}{dx}, \quad x^{n+2}\frac{d}{dx} \right).$$

The Lie algebra $\mathfrak{sl}(2)$ considered in [3, 5, 11] corresponds to $\mathfrak{l}_0$ generated by the natural action of SL(2) by linear-fractional (or Möbius) transformations (cf. [22]). Are there any other Lie subalgebras of Vect(M) for which the equivariant quantization procedure exists? On $\mathbb{R}$, all subalgebras that contain $\frac{d}{dx}$ were classified by Lie [20] (see also [9], [23]). Apart from $\mathfrak{l}_0$, they are (for a fixed $s \in \mathbb{R} \setminus \{0\}$):

$$\mathfrak{g}_0 \ = \ \text{Span}\left(\frac{d}{dx} \right), \quad \mathfrak{h}_0 \ = \ \text{Span}\left(\frac{d}{dx}, \quad e^{sx}\frac{d}{dx} \right), _{,1} \ = \ \text{Span}\left(\frac{d}{dx}, \quad x\frac{d}{dx} \right),$$

$$\mathfrak{k}_1 = \text{Span}\left(\frac{d}{dx}, \ \sin(sx)\frac{d}{dx}, \ \cos(sx)\frac{d}{dx} \right), \quad \mathfrak{k}_2 = \text{Span}\left(\frac{d}{dx}, \ \sinh(sx)\frac{d}{dx}, \ \cosh(sx)\frac{d}{dx} \right).$$

The Lie subalgebras $\mathfrak{l}_n$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$ correspond to various embeddings of $\mathfrak{sl}(2)$. The vector field $\frac{d}{dx}$ spans a commutative Lie algebra isomorphic to $\mathfrak{so}(2)$. We require that the Lie algebras

[*]E-mail:Bouarroudj.Sofiane@uaeu.ac.ae

U. A. E. University, Department of Mathematical Sciences, P. O. Box 17551, Al-Ain, U. A. E.

[†]E-mail:mohdayari@uaeu.ac.ae

Université du centre, institut supérieur d'informatique et de gestion de Kairouan, Tunisia.

we are dealing with contain $\mathfrak{so}(2)$. This property implies that the invariant differential operators we are studying are with constant coefficients (see sec. 3.1).

The existence of an equivariant quantization map induces a deformation of the $\operatorname{Vect}(M)$-action on the space of polynomials in the sense of the Nijenhuis-Richardson deformation theory [21]. As is well known, deformation of modules is related to the computation of a certain first cohomology group. In our case, the deformation becomes trivial when we restrict the action to $\mathfrak{sl}(2)$; besides, we deal only with a deformation that is expressed in terms of differential operators. Therefore, we will deal with the relative cohomology group

$$\mathrm{H}^1(\operatorname{Vect}(M), \mathfrak{sl}(2); \operatorname{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)), \tag{1.1}$$

where $\mathcal{F}_\lambda$ stands for the space of densities of weight λ and $\operatorname{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ stands for the space of differential operators on weighted densities.

Our first main result is the computation of (1.1) for $\mathfrak{sl}(2)$ realized as $\mathfrak{k}_1$ or $\mathfrak{k}_2$. Here, we only deal with cochains given by smooth maps. Using this result, we prove that there is no equivariant quantization map intertwining the actions of the Lie subalgebras $\mathfrak{k}_1$ and $\mathfrak{k}_2$.

Boniver and Mathonet [4] have investigated, in multi-dimensional case, all maximal Lie subalgebras of $\operatorname{Vect}(\mathbb{R}^n)$ for which the equivariant quantization exists.

2 Relative cohomology, statement of main results

In what follows, M stands for either $\mathbb{R}$ or S^1 and $\mathfrak{k} = \mathfrak{k}_1$ or $\mathfrak{k}_2$.

The space of tensor densities, denoted by $\mathcal{F}_\lambda$, is the space of sections of the line bundle $(T^*M)^{\otimes \lambda}$, where $\lambda \in \mathbb{R}$. This space has the following structure of a $\operatorname{Vect}(M)$-module: For any $\varphi(dx)^\lambda \in \mathcal{F}_\lambda$ and $X\frac{d}{dx} \in \operatorname{Vect}(M)$, we put

$$L^\lambda_{X\frac{d}{dx}}(\varphi(dx)^\lambda) = (X\varphi' + \lambda\,\varphi\operatorname{div} X)(dx)^\lambda. \tag{2.1}$$

Examples of such spaces are $\mathcal{F}_0 = C^\infty(M)$, $\mathcal{F}_{-1} = \operatorname{Vect}(M)$, and $\mathcal{F}_1 = \Omega^1(M)$.

A result of [3] is as follows:

$$\mathrm{H}^1(\operatorname{Vect}(M), \mathfrak{l}_0; \operatorname{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = \begin{cases} \mathbb{R} & \text{if } \mu - \lambda = \begin{cases} 2 & \text{and } \lambda \neq -\frac{1}{2} \\ 3 & \text{and } \lambda \neq -1 \text{ or } (\lambda, \mu) = \\ 4 & \text{and } \lambda \neq -\frac{3}{2} \end{cases} \\ & (-4, 1),\ (0, 5),\ \left(-\frac{5\pm\sqrt{19}}{2}, \frac{7\pm\sqrt{19}}{2}\right) \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

It is known ([16]) that

$$\mathrm{H}^1(\mathfrak{l}_0; \operatorname{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = \begin{cases} \mathbb{R} & \text{if } \mu - \lambda = 0 \\ \mathbb{R}^2 & \text{if } (\lambda, \mu) = \left(\frac{1-n}{2}, \frac{n+1}{2}\right), \quad \text{where } n \in \mathbb{N}\backslash\{0\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.1 *i)* $\mathrm{H}^1(\operatorname{Vect}(M), \mathfrak{k}; \operatorname{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = 0$ *for all λ and μ.*

ii) $\mathrm{H}^1(\mathfrak{k}; \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) \simeq \mathrm{H}^1(\mathrm{Vect}(M); \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$. *Moreover, if* $M = \mathbb{R}$, *we have (here* $\mathrm{H}^1_{\mathrm{diff}}$ *denotes the differential cohomology which means we deal only with cochains that are given by differential operators)*

$$
\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{k}; \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) =
\begin{cases}
\mathbb{R} & if \begin{cases} \mu - \lambda = 0, 2, 3 & or\ (\lambda, \mu) = \\ (-4, 1),\ (0, 5),\ \left(-\frac{5 \pm \sqrt{19}}{2}, \frac{7 \pm \sqrt{19}}{2}\right) \end{cases} \\
\mathbb{R}^2 & if\ (\lambda, \mu) = (0, 1) \\
0 & otherwise.
\end{cases}
$$

$$(2.4)$$

Remark 2.2 (i) The 1-cocycles that span $\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}); \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ were given in [10].

(ii) The cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(S^1); \mathfrak{k}; \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ cannot be described as in (2.4). The cocycles

$$
c_1(X\frac{d}{dx})(\phi(dx)^\lambda) = \mathrm{div}X\,\phi(dx)^{\mu-\lambda}, \quad and \quad c_2(X\frac{d}{dx})(\phi(dx)^\lambda) = X \cdot \phi(dx)^{\mu-\lambda},
$$

are not trivial and not cohomologous to each other for every pair (λ, μ) such that $\mu - \lambda = 0$.

The proof of Theorem 2.1 will be the subject of subsection 3.2 following the investigation of $\mathfrak{k}$-invariant bilinear differential operators.

3 Invariant operators and cohomology

3.1 $\mathfrak{k}$-invariant bilinear differential operators

First, we recall Grozman's result [13, 14] on the classification of bilinear differential operators on weighted densities invariant with respect to $\mathrm{Vect}(M)$:

i) Every zero-order operator $\mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\gamma+\lambda}$ is of the form

$$
\varphi(dx)^\gamma \otimes \psi(dx)^\lambda \mapsto \varphi \cdot \psi\,(dx)^{\gamma+\lambda}.
$$

ii) Every first-order operator $\mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\gamma+\lambda+1}$ is given by the Poisson bracket:

$$
\varphi\,(dx)^\gamma \otimes \psi\,(dx)^\lambda \mapsto \{\varphi, \psi\}\,(dx)^{\gamma+\lambda+1} \quad where \quad \{\varphi, \psi\} = \gamma\,\varphi\,\psi' - \lambda\,\varphi'\,\psi.
$$

iii) Every second-order operator $\mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\gamma+\lambda+2}$ is given by the composition of the de Rham operator and the Poisson bracket:

$$
\begin{aligned}
\varphi(dx)^\gamma \otimes \psi(dx)^\lambda &\mapsto d\{\varphi, \psi\}\,dx \quad for\ \lambda + \gamma + 1 = 0, \\
\varphi(dx)^\gamma \otimes \psi(dx)^\lambda &\mapsto \{d\,\varphi, \psi\}(dx)^{\lambda+2} \quad for\ \gamma = 0, \\
\varphi(dx)^\gamma \otimes \psi(dx)^\lambda &\mapsto \{\varphi, d\,\psi\}(dx)^{\gamma+2} \quad for\ \lambda = 0.
\end{aligned}
$$

iv) Every third-order invariant operator $\mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\gamma+\lambda+3}$ is given by the composition of the de Rham operator and the Poisson bracket. Moreover, there exists another

3

operator, now called Grozman operator, which is not given by this composition. The list is as follows:

$$\varphi \otimes \psi(dx)^{-2} \qquad\qquad \mapsto \quad d\{d\,\varphi, \psi\}\, dx \qquad\qquad\qquad \text{for } (\gamma, \lambda) = (0, -2)$$

$$\varphi(dx)^{-2} \otimes \psi \qquad\qquad \mapsto \quad d\{\varphi, d\,\psi\}\, dx \qquad\qquad\qquad \text{for } (\gamma, \lambda) = (-2, 0)$$

$$\varphi \otimes \psi \qquad\qquad\qquad \mapsto \quad \{d\,\varphi, d\,\psi\}(dx)^3 \qquad\qquad\quad \text{for } (\gamma, \lambda) = (0, 0)$$

$$\varphi(dx)^{-2/3} \otimes \psi(dx)^{-2/3} \quad \mapsto \quad \left(\begin{vmatrix} \varphi & \varphi''' \\ \psi & \psi''' \end{vmatrix} - \tfrac{3}{2} \begin{vmatrix} \varphi' & \varphi'' \\ \psi' & \psi'' \end{vmatrix} \right) (dx)^{5/3} \quad \text{for } (\gamma, \lambda) = (-\tfrac{2}{3}, -\tfrac{2}{3})$$

The list of $\mathfrak{l}$-invariant bilinear differential operators on weighted densities is a classical result of Gordan [12]. These operators are called "transvectants" (see [3, 11]). Their explicit expressions are given by

$$\mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \qquad \longrightarrow_{\gamma+\lambda+k}$$

$$\varphi(dx)^\gamma \otimes \psi(dx)^\lambda \quad \mapsto \qquad \sum_{i+j=k} (-1)^i \binom{k}{i} \frac{(2\gamma - i)\cdots(2\gamma - k + 1)}{(2\lambda - j)\cdots(2\lambda - k + 1)}\, \varphi^{(i)}\, \psi^{(j)}(dx)^{\gamma+\lambda+k}.$$

Now, we investigate bilinear differential operators that are invariant with respect to the Lie subalgebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$.

Theorem 3.1 *Up to a constant, the only $\mathfrak{l}$-invariant bilinear differential operators on tensor densities are: (i) the product of tensor densities, (ii) the Poisson bracket, (iii) the composition of the de Rham operator and Poisson bracket, (iv) the Grozman operator.*

Proof. Any bilinear differential operator $\mathcal{A} : \mathcal{F}_\gamma \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu$ can be expressed as

$$\mathcal{A}(\varphi, \psi) = \sum_{i+j \leq k} c_{i,j}\, \varphi^{(i)} \psi^{(j)}(dx)^\mu, \tag{3.1}$$

where $c_{i,j}$ are smooth functions and the superscript (i) stands for the i-th derivative. The $\mathfrak{l}$-invariance of the operator above reads as follows: for any $(\varphi(dx)^\gamma, \psi(dx)^\lambda) \in \mathcal{F}_\gamma \otimes \mathcal{F}_\lambda$ and $X\frac{d}{dx} \in \mathfrak{l}$, we have

$$L^\mu_{X\frac{d}{dx}} \mathcal{A}(\varphi, \psi) = \mathcal{A}(L^\gamma_{X\frac{d}{dx}} \varphi, \psi) + \mathcal{A}(\varphi, L^\lambda_{X\frac{d}{dx}} \psi). \tag{3.2}$$

The $\mathfrak{l}$-invariance (3.2) for $X\frac{d}{dx} = \frac{d}{dx}$ implies that $c'_{i,j} = 0$. Thus, $c_{i,j}$ are constants.

(a) For $k = 0$, the $\mathfrak{l}$-invariance (3.2) for $X\frac{d}{dx} = \sin(sx)\frac{d}{dx}$ and $\cos(sx)\frac{d}{dx}$ (or $\sinh(sx)\frac{d}{dx}$ and $\cosh(sx)\frac{d}{dx}$) is equivalent to the following equation

$$(-\gamma - \lambda + \mu)\, c_{0,0} = 0.$$

Therefore, $\mu = \gamma + \lambda$ and the corresponding operator is the multiplication operator given as in Grozman's list.

(b) For $k = 1$, the $\mathfrak{l}$-invariance is equivalent to the following system

$$\begin{aligned} (1 + \gamma + \lambda - \mu)\, c_{1,0} &= 0, & (1 + \gamma + \lambda - \mu)\, c_{0,1} &= 0, \\ (\gamma + \lambda - \mu)\, c_{0,0} &= 0, & \gamma\, c_{0,1} + \lambda\, c_{1,0} &= 0. \end{aligned} \tag{3.3}$$

If $1 + \gamma + \lambda - \mu \neq 0$, then $c_{1,0} = c_{0,1} = 0$. Thus the operator is 0-order, but this case has already been studied in Part (a). Therefore, $1 + \gamma + \lambda - \mu = 0$. The third equation implies that $c_{0,0} = 0$ which implies that the operator (3.1) is homogeneous. Now the last equation admits a solution for all λ and γ. The corresponding operator coincides (up to a factor) with the Poisson bracket.

(c) For $k = 2$, the $\mathfrak{k}$-invariance is equivalent to the system (3.3) and the following system:

$$\lambda\, c_{1,1} + (1 + 2\gamma)\, c_{2,0} \;=\; 0, \quad \gamma\, c_{1,1} + (1 + 2\lambda)\, c_{0,2} \;=\; 0, \qquad \gamma\, c_{2,0} + \lambda\, c_{0,2} \;=\; 0,$$

$$(2 + \gamma + \lambda - \mu)\, c_{1,1} \;=\; 0, \quad (2 + \gamma + \lambda - \mu)\, c_{2,0} \;=\; 0, \quad (2 + \gamma + \lambda - \mu)\, c_{0,2} \;=\; 0.$$

If $2 + \gamma + \lambda - \mu \neq 0$, then the constants $c_{2,0} = c_{0,2} = c_{1,1} = 0$. In this case the operator (3.1) becomes 1-order which has already been studied in Part (b). If $2 + \gamma + \lambda - \mu = 0$, then the constants $c_{1,0} = c_{0,1} = c_{0,0} = 0$ and the operator becomes homogeneous. The only values of λ and γ for which the system is consistent are given as in Part iii) of Grozman's list.

(d) For $k = 3$, we proceed as before and the operator (3.2) is given as in Grozman's list.

(e) Let us prove that for $k \geq 4$ there are no operators invariant with respect to the Lie algebra $\mathfrak{k}$. We proceed by induction; of course, a direct proof can be checked for $k = 4$. The invariance property implies that the coefficients of the components $X'\varphi^{(s)}\psi^{(k-s)}$ which should vanish is given by the system

$$(k + \gamma + \lambda - \mu)\, c_{s,k-s} \quad \text{for} \quad s = 0, \ldots, k.$$

If $k + \gamma + \lambda - \mu \neq 0$, then all the constants $c_{s,k-s} = 0$. Thus, the operator (3.1) is $(k-1)$-order and the induction assumption assures the non-existence. Suppose then that $k + \gamma + \lambda - \mu = 0$. The invariance implies that the components of $X'\varphi^{(s)}\psi^{(t)}$, where $s + t < k$, are given by

$$(s + t + \gamma + \lambda - \mu)\, c_{s,t} = 0 \quad \text{for} \quad s + t = 0, \ldots, k - 1.$$

Thus, all $c_{s,t} = 0$ for all $s + t < k$. Hence the operator (3.1) becomes homogeneous, namely

$$\sum_{i+j=k} c_{i,j}\, \varphi^{(i)}\psi^{(j)}(dx)^{\gamma+\lambda+k}.$$

Now, the invariance gives a system equivalent to the system coming from the invariance property with respect to the Lie algebra $\mathrm{Vect}(M)$. The result follows from Grozman's classification.

$\blacksquare$

3.2 Proof of Theorem 2.1.

Lemma 3.2 *[18] Every 1-cocycle c on $\mathrm{Vect}(M)$ with values in $\mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ is differentiable.*

Lemma 3.3 *Every 1-cocycle c on $\mathrm{Vect}(M)$ that vanishes on $\mathfrak{k}$ is necessarily $\mathfrak{k}$-invariant.*

Proof. The 1-cocycle property reads

$$L_{X\frac{d}{dx}} c(Y, A) - L_{Y\frac{d}{dx}} c(X, A) = c([X, Y], A),$$

for any $X\frac{d}{dx}, Y\frac{d}{dx} \in \mathrm{Vect}(M)$ and $A \in \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. As c vanishes on $\mathfrak{k}$, then

$$L_{X\frac{d}{dx}} c(Y, A) = c([X, Y], A) \quad \text{for any } X\frac{d}{dx} \in \mathfrak{k}.$$

This is just the $\mathfrak{k}$-invariance property.

$\blacksquare$

According to Theorem 3.1, there are only two $\mathfrak{k}$-invariant differential operators

$$\mathrm{Vect}(M) \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu,$$

and they are given by the Poisson bracket $\{\cdot, \cdot\}$ for $\gamma = -1$ and the operator $\{\cdot, d\}$ for $(\gamma, \lambda) = (-1, 0)$. Let us describe under what conditions the operator $\{\cdot, d\}$ is a 1-cocycle. For any $\psi \in \mathcal{F}_0$ and $X\frac{d}{dx}, Y\frac{d}{dx} \in \mathrm{Vect}(M)$, we have

$$\{[X, Y], d\,\psi\} = L_{X\frac{d}{dx}}\{Y, d\,\psi\} - \{Y, d\,L_{X\frac{d}{dx}}\psi\} - L_{Y\frac{d}{dx}}\{X, d\,\psi\} + \{X, d\,L_{Y\frac{d}{dx}}\psi\}.$$

The cocycle condition implies that the coefficients $c_{2,0}, c_{0,2}$ and $c_{1,1}$ of the operator $\{\cdot, d\}$ satisfy the system

$$(\mu - 1)\, c_{2,0} = 0, \quad c_{0,2} = 0, \quad (2 - \mu)\, c_{0,2} = 0.$$

Therefore, the operator $\{\cdot, d\}$ is never a 1-cocycle. The same holds for the Poisson bracket $\{\cdot, \cdot\}$, and therefore we have just proved that $\mathrm{H}^1(\mathrm{Vect}(M), \mathfrak{k}; \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = 0$ and (i) is proven.

To prove ii), let c be a 1-cocycle on $\mathrm{Vect}(M)$ with value in $\mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$, and denote by $\widehat{c}$ its restriction to $\mathfrak{k}$. We will prove that the map $c \mapsto \widehat{c}$ is an isomorphism. Let c_1 and c_2 be two 1-cocycles such that $\widehat{c}_1 = \widehat{c}_2$. It follows that the difference $c_1 - c_2$, which is also a 1-cocycle, vanishes on $\mathfrak{k}$. Part i) of the Theorem ensures that it is a coboundary, namely

$$c_1(X) - c_2(X) = L_{X\frac{d}{dx}} \circ A - A \circ L_{X\frac{d}{dx}} \quad \text{for all } X\frac{d}{dx} \in \mathrm{Vect}(M),$$

where A is a certain operator in $\mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. Thus, $c_1 \equiv c_2$.

Let s be a 1-cocycle on $\mathfrak{k}$ with values in $\mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. We will prove that there exists a 1-cocycle on $\mathrm{Vect}(M)$, say c, such that $\widehat{c} = s$. To do that, we write the 1-cocycle s as follows. For any $X\frac{d}{dx} \in \mathrm{Vect}(M)$ and $\psi(dx)^\lambda \in \mathcal{F}_\lambda$, we put (where $c_{i,j}$ are smooth functions)

$$s(X, \psi) = \sum_{i,j} c_{i,j}\, X^{(i)}\, \psi^{(j)}(dx)^\mu.$$

The cocycle property for $X\frac{d}{dx} = \cos(sx)\frac{d}{dx}$ (or $\cosh(sx)\frac{d}{dx}$) is equivalent to the following system

$$i(i + j - 1 - \mu + \lambda)\, c_{i,j} - (j + \lambda i)\binom{j+i-1}{i} c_{1,j-1+i} \;=\; 0 \quad \text{for } i + j \le k \text{ and } i > 1,$$

$$\frac{j-i}{j}\binom{j+i-1}{i} c_{i+j-1,l} + \left(\binom{l+i-1}{i} + \lambda\binom{l+i-1}{i-1}\right) c_{j,l-1+i}$$

$$-\left(\binom{l+j-1}{j} + \lambda\binom{l+j-1}{j-1}\right) c_{i,l-1+j} \;=\; 0 \quad \text{for } l + i + j \le k + 1 \text{ and } j > i > 1,$$

$$(\mu - i - \lambda)\, c_{0,i} - c'_{1,i} \;=\; 0 \quad \text{for } 0 < i \le k,$$

$$(j - 1 - \mu + l + \lambda)\, c_{j,l} - \left(\binom{l+j-1}{j} + \lambda\binom{l+j-1}{j-1}\right) c_{1,l-1+j} \;=\; 0 \quad \text{for } j + l \le k.$$

6

Now, a direct computation proves that the operator defined by $\sum_{i,j} c_{i,j} X^{(i)} \psi^{(j)} (dx)^\mu$, where $c_{i,j}$ are as above, satisfies the 1-cocycle property on the entire $\text{Vect}(M)$. In other words, any 1-cocycle on $\mathfrak{k}$ is certainly a 1-cocycle on $\text{Vect}(M)$.

The proof of (ii) for $M = \mathbb{R}$ follows from computations of $\text{H}^1(\text{Vect}(\mathbb{R}), \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ due to Feigin-Fuchs [10]. Our Theorem is proven.

4 Quantization equivariant with respect to $\mathfrak{k}$

The modules of linear differential operators on the spaces of tensor densities on a smooth manifold are classical objects (see [24]). Recently, they have been intensively studied in a series of papers (see [1, 2, 3, 6, 8, 11, 17, 18]). Let $\mathcal{D}^k_{\lambda,\mu}$ be the space of order k differential operators $A : \mathcal{F}_\lambda \to \mathcal{F}_\mu$ endowed with the $\text{Vect}(M)$-module structure by the formula

$$L^{\lambda,\mu}_{X\frac{d}{dx}}(A) = L^\mu_{X\frac{d}{dx}} \circ A - A \circ L^\lambda_{X\frac{d}{dx}}. \tag{4.1}$$

Let $\delta = \mu - \lambda$. For first-order differential operators, it was proven in [11] that there exists a map

$$\mathcal{F}_{\delta-1} \longrightarrow \mathcal{D}^1_{\lambda,\mu} \qquad a_1(dx)^{\delta-1} \mapsto \left(a_1 \frac{d}{dx} + \frac{\lambda}{1-\delta} a_1' \right) dx^\delta$$

equivariant with respect to $\text{Vect}(M)$. However, for second-order differential operators, it was proven in [5, 11] that there exists a map

$$\mathcal{F}_{\delta-2} \longrightarrow \mathcal{D}^2_{\lambda,\mu} \qquad a_2(dx)^{\delta-2} \mapsto \left(a_2 \frac{d^2}{dx^2} + \frac{1+2\lambda}{2-\delta} a_2' \frac{d}{dx} + \frac{\lambda(1+2\lambda)}{(2-\delta)(3-2\delta)} a_2'' \right) dx^\delta$$

equivariant with respect to $\mathfrak{l}_0$ but there is no map $\mathcal{F}_{\delta-2} \longrightarrow \mathcal{D}^2_{\lambda,\mu}$ equivariant with respect to the whole Lie algebra $\text{Vect}(M)$. For $\mathfrak{k}$, we have the following

Theorem 4.1 *A differential operator*

$$Q : \mathcal{F}_{\delta-2} \oplus \mathcal{F}_{1-\delta} \oplus \mathcal{F}_\delta \longrightarrow \mathcal{D}^2_{\lambda,\mu}$$

equivariant with respect to $\mathfrak{k}$ exists only for either $\mu = 1$ and $\lambda \neq -1$, or $\mu \neq 2$ and $\lambda = 0$.

Proof. Assume the contrary: Let the map Q exist. We have, therefore, a diagram

$$
\begin{array}{ccc}
\mathcal{F}_{\delta-2} \oplus \mathcal{F}_{\delta-1} \oplus \mathcal{F}_\delta & \xrightarrow{\;Q\;} & \mathcal{D}^2_{\lambda,\mu} \\
\downarrow & & \downarrow{\scriptstyle L^{\lambda,\mu}_{X\frac{d}{dx}}} \\
\mathcal{F}_{\delta-2} \oplus \mathcal{F}_{\delta-1} \oplus \mathcal{F}_\delta & \xrightarrow{\;Q\;} & \mathcal{D}^2_{\lambda,\mu}
\end{array}
$$

that by our assumption commutes once the action is restricted to $\mathfrak{k}$. Hence the map

$$Q^{-1} \circ L^{\lambda,\mu}_{X\frac{d}{dx}} \circ Q,$$

defines a new action on the space $\mathcal{F}_{\delta-2} \oplus \mathcal{F}_{\delta-1} \oplus \mathcal{F}_\delta$. Thus, there exists a map c such that

$$Q^{-1} \circ L^{\lambda,\mu}_{X\frac{d}{dx}} \circ Q = (L^{\delta-2}_{X\frac{d}{dx}}, L^{\delta-1}_{X\frac{d}{dx}}, L^\delta_{X\frac{d}{dx}}) + c(X),$$

where $L^\delta_{X\frac{d}{dx}}$ is the action (2.1). It is easy to see that c defines a 1-cocycle on $\text{Vect}(M)$ with values in $\text{End}_{\text{diff}}(\mathcal{F}_{\delta-2} \oplus \mathcal{F}_{\delta-1} \oplus \mathcal{F}_\delta)$, vanishing on $\mathfrak{k}$. By Theorem 3.1, the map c is identically zero, not just a coboundary. Thus, the map Q is not only $\mathfrak{k}$-equivariant but also $\text{Vect}(M)$-equivariant. But by [11] there is no quantization map equivariant with respect to the whole Lie algebra $\text{Vect}(M)$ except for $\mu = 1$ and $\lambda \neq -1$, or $\mu \neq 2$ and $\lambda = 0$. ∎

Conclusion

Even though the Lie subalgebras $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are of finite dimension, the (differential) equivariant quantization map does not exist. We have shown in this paper that the existence is related to the nature of the vector fields that generate a Lie subalgebra, rather than the number of generators. We believe that the dimension of a Lie subalgebra will determine only the uniqueness of the quantization map – whenever it exists – as already pointed out in [7].

Acknowledgments

References

[1] Bouarroudj S, Cohomology of the vector fields Lie algebras on S^1 acting on bilinear differential operators, *Internat. Jour. Geom. Methods. Mod. Phys.* **2,** no.1, (2005), 23–40.

[2] Bouarroudj S and Gargoubi H, Projectively Invariant Cocycles of Holomorphic Vector Fields on an Open Riemann Surface. *Tokyo J. Math.* 25, (2002), no.1, 33–40.

[3] Bouarroudj S and Ovsienko V, Three cocycles on Diff(S^1) generalizing the Schwarzian derivative, *Int. Math. Res. Not.*, (1998), No.1, 25–39.

[4] Boniver F and Mathonet P, Maximal subalgebras of vector fields for equivariant quantizations, *Jour. Math. Phys.* **42** (2), (2001), 582–589.

[5] Cohen P, Manin Yu and Zagier D, Automorphic pseudodifferential operators, *Algebraic aspects of integrable systems,* 17–47, Progr. Nonlinear Differential Equations Appl., 26, Birkhäuser Boston, Boston, MA, 1997.

[6] Duval C and Ovsienko V, Space of second order linear differential operators as a module over the Lie algebra of vector fields, *Adv. in Math.* **132:2**, (1997), 316–333.

[7] Duval C, Lecomte P and Ovsienko V, Conformally equivariant quantization: existence and uniqueness. *Ann. Inst. Fourier* 49, No.6, (1999), 1999-2029.

[8] Duval C and Ovsienko V, Conformally equivariant quantum Hamiltonians. *Selecta Math. (N.S.)* **7**, no. 3, (2001), 291–320.

[9] Eastwood M and Slovàk J, Preferred parametrizations on homogenous curves, `math.DG`/0311456.

[10] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras,* Contemp. Soviet Math., Consultants Bureau, New york, 1986.

[11] Gargoubi H, Sur la géométrie de l'espace des opérateurs différentiels linéaires sur $\mathbb{R}$. Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, (2000), 21–47.

[12] Gordan P, *Invariantentheorie,* Teubner, Leipzig, 1887.

[13] Grozman P Ya, Classification of bilinear invariant operators over tensor fields, *Functional Anal. Appl.,* 14:2, (1980), 127–128.

[14] Grozman P Ya, Invariant bilinear differential operators, Preprint ESI 1114 (2001), `math.RT`/0509562.

[15] Janson S and Peetre J, A new generalization of Hankel operators (the case of higher weights), Math. Nachr. **132**, (1987), 313-328.

[16] Lecomte P B A, On the cohomology of $sl(n+1, \mathbb{R})$ acting on differential operators and $sl(n+1, \mathbb{R})$-equivariant symbols, *Indag. Math. NS.* 11 (1), (2000), 95–114.

[17] Lecomte P B A, Mathonet P and Tousset E, *Comparison of some modules of the Lie algebra of vector fields*, Indag. Math., N.S., **7**:4, (1996), 461–471.

[18] Lecomte P B A and Ovsienko V, Cohomology of vector fields Lie algebras, *Compositio Mathematica.* **124:** No.1, (2000), 95–110.

[19] Lecomte P B A and Ovsienko V, Projectively invariant symbol calculus, *Lett. Math. Phy.* **49** (3), (1999), 173–196.

[20] Lie S, Theorie der transformationsgruppen, *Math. Ann.* **16**, (1880), 441–528.

[21] Nijenhuis A and Richardson R W, Deformation of homomorphisms of Lie algebras, Bull. AMS **73** (1967) 175–179.

[22] Olver P, *Applications of Lie groups to differential equations.* Springer Verlag.

[23] Post G, On the structure of transitively differential algebras, *J. Lie Theory* **11**, No. 1, (2001), 111–128.

[24] Wilczynski E J, *Projective differential geometry of curves and ruled surfaces*, Teubner, Leipzig, 1906.

[25] Woodhouse N M J, *Geometric quantization.* Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.